and $T_e = 3500\,^\circ\mathrm{K}$ with the pressure and heavy particle temperature being 1 torr and $1000\,^\circ\mathrm{K}$, respectively. The value of $a \approx 10^{-3}$. Using the atomic constants for the CI 2479 line given in Ref. 8, Eq. (3) gives $\kappa_0{}^*L = -1.46$ for Run 5. Assuming a path length $L = 5$ cm, the carbon atom density is $\approx 3.3 \times 10^{14}\ \mathrm{cm}^{-3}$ while $N_l{}^* = -10^{12}\ \mathrm{cm}^{-3}$. Assuming the low states to be in partial equilibrium among themselves[1] at an effective excitation temperature T_e, we find $(N_u/g_u)/(N_l/g_l) = 2 \times 10^2$ for Run 5. It should be noted that the measured A_L values are averaged across the jet and local values may be more negative.

Conclusion

Experimental measurements have indicated that a substantial population inversion can be produced in a high-pressure arc heated carbon plasma expanding through a nozzle for the carbon I 2478.6 Å line.

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Dynamical Torsion Theory of Rods Deduced from the Equations of Linear Elasticity

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Introduction

SEARCHING through the last two decades of the literature of mechanics, it is a common occurrence to find the extensional and flexural equations of plates and rods derived by suitably averaging the equations of linear elasticity through the thickness (for plates) or over the cross section (for rods).[1,2] A similar technique for finding the torsional equations of a rod appears to be missing from the literature of mechanics.† The purpose of this Note is to provide such a formulation which is of interest for two reasons. Firstly it is of intrinsic interest to show the intimate relation of the linear equations of elasticity to the dynamical torsion equation and secondly it is of interest to note that, since this technique can be easily extended to average all nonlinear equations of motion, the averaging technique offers a simple alternative to the variational principle as a means of generating approximate equations of motion. This use of the averaging technique has been employed by the author to solve several classes of problems under initial stress and will be the subject of several forthcoming publications.

The present analysis is presented in two parts, the first part being a derivation of the classical torsion equation and the second part being a derivation of the torsion equation with warping effects included.

Mathematical and Physical Preliminaries

Introducing the elastic constants E, $G_{13} = G_{12} = G$, $G_{23} = G_{32} \to \infty$, the warping function $\phi(x_1, x_2, x_3)$, and the twist angle $\theta(x_1, t)$ and assuming the following displacements[4]

$$u_1 = \phi(\partial\theta/\partial x_1) \tag{1}$$

$$u_2 = -x_3\theta \tag{2}$$

$$u_3 = x_2\theta \tag{3}$$

the nonzero strains are given by,

$$\varepsilon_{11} = \partial u_1/\partial x_1 = \partial/\partial x_1(\phi(\partial\theta/\partial x_1)) \tag{4}$$

$$\varepsilon_{12} = \tfrac{1}{2}(\partial u_1/\partial x_2 + \partial u_2/\partial x_1) = \tfrac{1}{2}(\partial\theta/\partial x_1)(\partial\phi/\partial x_2 - x_3) \tag{5}$$

$$\varepsilon_{13} = \tfrac{1}{2}(\partial u_1/\partial x_3 + \partial u_3/\partial x_1) = \tfrac{1}{2}(\partial\theta/\partial x_1)(\partial\phi/\partial x_3 + x_2) \tag{6}$$

and the corresponding stresses‡ are given by,

$$\sigma_{11} = E(\partial/\partial x_1)(\phi(\partial\theta/\partial x_1)) \tag{7}$$

$$\sigma_{12} = G(\partial\theta/\partial x_1)(\partial\phi/\partial x_2 - x_3) \tag{8}$$

$$\sigma_{13} = G(\partial\theta/\partial x_1)(\partial\phi/\partial x_3 + x_2) \tag{9}$$

The equations of linear elasticity

$$\partial\sigma_{ij}/\partial x_i + x_j = \rho\ddot{u}_j \qquad \text{(10a), (10b), (10c)}$$

are now suitably averaged in the two following sections to obtain the desired equations of motion.

Classical Torsion Equation

For the classical case the averaged torque equation is formed by adding the averages (over the cross section) of Eq. (10b) multiplied by $-x_3$ and Eq. (10c) multiplied by x_2. Hence putting Eqs. (2) and (8) into Eq. (10b), multiplying by $-x_3$ and averaging over the cross section (carrying out one integration by parts) yields

$$\frac{\partial}{\partial x_1}\left\{G\frac{\partial\theta}{\partial x_1}\int\!\!\int\left(-x_3\frac{\partial\phi}{\partial x_2} + x_3^2\right)dx_2\,dx_3\right\} - \int_{x_3^-}^{x_3^+} x_3\Big[\sigma_{22}\Big]_{x_2^-}^{x_2^+} dx_3 - \int_{x_2^-}^{x_2^+}\Big[x_3\sigma_{32}\Big]_{x_3^-}^{x_3^+} dx_2 + \int\!\!\int\sigma_{32}\,dx_2\,dx_3 - \int\!\!\int x_3 X_2\,dx_2\,dx_3 = \ddot{\theta}\int\!\!\int\rho x_3^2\,dx_2\,dx_3 \tag{11}$$

Similarly putting Eqs. (3) and (9) into Eq. (10c), multiplying by x_2 and averaging over the cross section (carrying out one integration by parts) yields

$$\frac{\partial}{\partial x_1}\left\{G\frac{\partial\theta}{\partial x_1}\int\!\!\int\left(x_2\frac{\partial\phi}{\partial x_3} + x_2^2\right)dx_2\,dx_3\right\} + \int_{x_3^-}^{x_3^+}\Big[x_2\sigma_{23}\Big]_{x_2^-}^{x_2^+} dx_3 - \int\!\!\int\sigma_{23}\,dx_2\,dx_3 + \int_{x_2^-}^{x_2^+} x_2\Big[\sigma_{33}\Big]_{x_3^-}^{x_3^+} dx_2 + \int\!\!\int x_2 X_3\,dx_2\,dx_3 = \ddot{\theta}\int\!\!\int\rho x_2^2\,dx_2\,dx_3 \tag{12}$$

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† It must be noted that Warner[3] attempted such a technique in 1965 but his method produced the torsional (shear) wave equation rather than the desired torsional equations of motion.

‡ In general all the stresses are nonzero. However we will need explicit forms for only the stresses given in Eqs. 7, 8, and 9. This is because the other stresses either cancel in pairs in the analysis or appear only in terms evaluated at the boundary.

Noting that,

$$\int\int\left(x_2\frac{\partial\phi}{\partial x_3} - x_3\frac{\partial\phi}{\partial x_2} + x_2^2 + x_3^2\right)dx_2\,dx_3 \equiv J(x_1)$$

$$= \text{torsion const} \qquad (13)$$

and

$$\int\int\rho(x_2^2 + x_3^2)dx_2\,dx_3 \equiv I(x_1)$$

$$= \text{polar mass moment of inertia} \qquad (14)$$

the addition of the averaged torque Eqs. (11) and (12) yields the torsional equation of motion

$$\frac{\partial}{\partial x_1}\left(GJ\frac{\partial\theta}{\partial x_1}\right) + \tilde{t} = I\ddot{\theta} \qquad (15)$$

where the torque per unit length $\tilde{t}$ is defined as the sum of the torque boundary terms in Eqs. (11) and (12). That is

$$\tilde{t} = \int_{x_3^-}^{x_3^+}[x_2\sigma_{23}]_{x_2^-}^{x_2^+}dx_3 - \int_{x_2^-}^{x_2^+}[x_3\sigma_{32}]_{x_3^-}^{x_3^+}dx_2 + \int_{x_2^-}^{x_2^+}x_2[\sigma_{33}]_{x_3^-}^{x_3^+}dx_2$$

$$- \int_{x_3^-}^{x_3^+}x_3[\sigma_{22}]_{x_2^-}^{x_2^+}dx_3 + \int\int(x_2X_3 - x_3X_2)dx_2\,dx_3 \qquad (16)$$

It is readily seen by inspecting each term of Eq. (16) that they are terms which contribute to the torque applied at the boundary. Additionally, the torque T exerted at any cross section x_1 is given by

$$T = \int\int(x_2\sigma_{13} - x_3\sigma_{12})dx_2\,dx_3$$

$$= G\frac{\partial\theta}{\partial x_1}\int\int\left(x_2\frac{\partial\phi}{\partial x_3} - x_3\frac{\partial\phi}{\partial x_2} + x_2^2 + x_3^2\right)dx_2\,dx_3 \qquad (17)$$

Using Eq. (13), Eq. (17) then becomes

$$T = GJ(\partial\theta/\partial x_1) \qquad (18)$$

Hence the desired dynamical torsion equation of motion and the torque relation are given by Eqs. (15) and (18), respectively.

Torsion Equation with Warping ¶

The case of torsion with warping requires that Eq. (10a) also be suitably averaged. To this end Eq. (10a) is multiplied by $-\phi$, differentiated with respect to x_1 and then averaged over the cross section to yield

$$-E\Gamma\frac{\partial^4\theta}{\partial x_1^4} - E\Gamma_1\frac{\partial^3\theta}{\partial x_1^3} - E\Gamma_2\frac{\partial^2\theta}{\partial x_1^2} - E\Gamma_3\frac{\partial\theta}{\partial x_1}$$

$$+ \frac{\partial}{\partial x_1}\left\{G\frac{\partial\theta}{\partial x_1}\int\int\left[\left(\frac{\partial\phi}{\partial x_2} - x_3\right)\frac{\partial\phi}{\partial x_2} + \left(\frac{\partial\phi}{\partial x_3} + x_2\right)\frac{\partial\phi}{\partial x_3}\right]dx_2\,dx_3\right\}$$

$$+ 2\rho\Gamma_1\frac{\partial\ddot{\theta}}{\partial x_1} + \rho\Gamma\frac{\partial^2\ddot{\theta}}{\partial x_1^2} - \int\int\frac{\partial}{\partial x_1}(\phi X_1)dx_2\,dx_3$$

$$- \int_{x_3^-}^{x_3^+}\frac{\partial}{\partial x_1}[\phi\sigma_{21}]_{x_2^-}^{x_2^+}dx_3 - \int_{x_2^-}^{x_2^+}\frac{\partial}{\partial x_1}[\phi\sigma_{31}]_{x_3^-}^{x_3^+}dx_2 = 0 \qquad (19)$$

where

$$\Gamma = \int\int\phi^2\,dx_2\,dx_3 \qquad (20)$$

$$\Gamma_1 = 4\int\int\phi\frac{\partial\phi}{\partial x_1}dx_2\,dx_3 \qquad (21)$$

$$\Gamma_2 = \int\int\left[2\left(\frac{\partial\phi}{\partial x_1}\right)^2 + 3\phi\frac{\partial^2\phi}{\partial x_1^2}\right]dx_2\,dx_3 \qquad (22)$$

$$\Gamma_3 = \int\int\frac{\partial}{\partial x_1}\left(\phi\frac{\partial^2\phi}{\partial x_1^2}\right)dx_2\,dx_3 \qquad (23)$$

Adding Eq. (19) to Eqs. (11) and (12) yields the desired equation of motion

$$-E\Gamma\frac{\partial^4\theta}{\partial x_1^4} - E\Gamma_1\frac{\partial^3\theta}{\partial x_1^3} + E\Gamma_2\frac{\partial^2\theta}{\partial x_1^2} - E\Gamma_3\frac{\partial\theta}{\partial x_1} + \frac{\partial}{\partial x_1}\left(GC\frac{\partial\theta}{\partial x_1}\right)$$

$$+ 2\rho\Gamma_1\frac{\partial\ddot{\theta}}{\partial x_1} + \rho\Gamma\frac{\partial^2\ddot{\theta}}{\partial x_1^2} + t^* = I\ddot{\theta} \qquad (24)$$

where

$$C = \int\int\left[\left(\frac{\partial\phi}{\partial x_3} + x_2\right)^2 + \left(\frac{\partial\phi}{\partial x_2} - x_3\right)^2\right]dx_2\,dx_3$$

and

$$t^* = \tilde{t} - \int\int\frac{\partial}{\partial x_1}(\phi X_1)dx_2\,dx_3 - \int_{x_3^-}^{x_3^+}\frac{\partial}{\partial x_1}[\phi\sigma_{21}]_{x_2^-}^{x_2^+}dx_3$$

$$- \int_{x_2^-}^{x_2^+}\frac{\partial}{\partial x_1}[\phi\sigma_{31}]_{x_3^-}^{x_3^+}dx_2$$

Equation (24) is a generalized version (ϕ has been allowed an x_1 dependence) of the torsion equation with warping which appears in the literature. If ϕ is not a function of x_1, Eq. (24) reduces to its usual form

$$-E\Gamma\frac{\partial^4\theta}{\partial x_1^4} + \frac{\partial}{\partial x_1}\left(GC\frac{\partial\theta}{\partial x_1}\right) + \rho\Gamma\frac{\partial^2\ddot{\theta}}{\partial x_1^2} + t^* = I\ddot{\theta} \qquad (25)$$

Furthermore, the torque T at any x_1 cross section is given by

$$T = \int\int(x_2\sigma_{13} - x_3\sigma_{12} - \phi\nabla\cdot\boldsymbol{\sigma})dx_2\,dx_3 \qquad (26)$$

where

$$\boldsymbol{\sigma} = \sigma_{11}\boldsymbol{i}_1 + \sigma_{12}\boldsymbol{i}_{12} + \sigma_{13}\boldsymbol{i}_{13} \qquad (27)$$

and

$$\nabla = (\partial/\partial x_1)\boldsymbol{i}_1 + (\partial/\partial x_2)\boldsymbol{i}_2 + (\partial/\partial x_3)\boldsymbol{i}_3 \qquad (28)$$

Therefore,

$$T = \int\int\left(x_2\sigma_{13} - x_3\sigma_{12} - \phi\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\phi}{\partial x_2}\sigma_{12} + \frac{\partial\phi}{\partial x_3}\sigma_{13}\right)dx_2\,dx_3$$

$$= \int\int\left\{G\frac{\partial\theta}{\partial x_1}\left[\left(\frac{\partial\phi}{\partial x_2} - x_3\right)^2 + \left(\frac{\partial\phi}{\partial x_3} + x_2\right)^2\right]\right.$$

$$\left. - E\phi^2\frac{\partial^3\theta}{\partial x_1^3}\right\}dx_2\,dx_3 \qquad (29)$$

and in final form

$$T = GC(\partial\theta/\partial x_1) - E\Gamma(\partial^3\theta/\partial x_1^3) \qquad (30)$$

Hence the desired dynamical torsion equation of motion and the torque relation are given by Eqs. (24) and (30), respectively.

Conclusions

As demonstrated in the last two sections the averaging technique presents a simple but powerful alternative to the variational principle for formulating approximate equations of motion. The two principal virtues of the averaging technique are 1) that it does not require the knowledge of an elaborate mathematical technique (the calculus of variations) to formulate results and 2) that it provides some physical insight into what is happening at each stage of development of the desired equations; the

¶ The author is indebted to G. L. Anderson for suggesting the method of approach used in this section.

same cannot be said for the manipulations required in carrying out the variational procedure.

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Method for Constructing a Full Modal Damping Matrix from Experimental Measurements

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THE normal mode method is widely used for dynamic analysis of linear structures. By enabling the equations of motion to be written in terms of modal coordinates, solutions are more readily determined. Fortunately, structural damping tends to be small so that the classical undamped modes have a useful physical interpretation. It is common practice to introduce damping only after the equations have been transformed to modal coordinates. In this case, viscous damping is typically assumed and the modal damping matrix is taken to be diagonal. The elements along the diagonal are related to the percent of critical damping for each mode, while the rest of the matrix is assumed to be null implying that the modes are not coupled by damping forces in the structure.

If linear viscous damping is assumed at the outset, there are in fact only certain special cases for which the damping matrix is diagonalized by a transformation to modal coordinates. One case is that of proportional damping where the damping matrix is a linear combination of the mass and stiffness matrices.[1] The complementary class is referred to as nonproportional damping which in general does not diagonalize, although Caughey has identified subsets of this class which do.[2] Nevertheless, there seems to be no strong physical basis for any of the special cases so that in all probability the off-diagonal coupling terms will exist. In general they will be of the same order as the diagonal terms as will subsequently be shown.

Although justification may be found for neglecting these terms in some analyses, there are certainly times when this is inappropriate. For example when modal synthesis is employed to combine substructure characteristics in deriving the equations of motion for a complete structure, and linear viscous damping is taken to represent the dissipative mechanism of the structure, the full modal damping matrices are required for each substructure. Since the off-diagonal terms are likely to be of the same order as the diagonal ones, they too will influence the modal damping being computed for the complete structure. Although objections have been raised over the use of viscous damping,[3] it appears doubtful that the use of more general damping models can be justified yet, since considerably more information is required to define their parameters. The proper use of viscous damping should be fully explored first since it represents the simplest approach. A method for measuring the off-diagonal terms of the modal damping matrix is the subject of this Note. Since this entails more work, the likelihood of proportional damping should first be established.

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A Necessary Condition for the Existence of Proportional Damping

If proportional damping is presumed to exist, the homogeneous equations of motion can be written in the form

$$m\ddot{x} + (\alpha m + \beta k)\dot{x} + kx = 0 \tag{1}$$

where x is a vector of generalized displacements, m and k are, respectively, the mass and stiffness matrices of the structure, and α and β are scalar constants. Solutions will be of the form

$$x(t) = \phi_{R_j} e^{\lambda_j t} = \phi_{R_j} e^{(\sigma_j + i\omega_j)t} \tag{2}$$

where ϕ_{R_j} denotes the real modal displacement vector of the jth mode and $\lambda_j = \sigma_j + i\omega_j$ is the corresponding complex eigenvalue. The quantities ω_j and $-\sigma_j$ are interpreted as the damped modal frequency and decay rate. The undamped frequency ω_{0_j} and critical damping ratio ζ_j are given in terms of σ_j and ω_j by

$$\omega_{0_j} = (\sigma_j^2 + \omega_j^2)^{1/2}, \quad \zeta_j = -\sigma_j/(\sigma_j^2 + \omega_j^2)^{1/2}$$

In the case of proportional damping, it may be shown that ζ_j and ω_{0_j} for each mode are related by

$$\zeta_j = \alpha/\omega_{0_j} + \beta\omega_{0_j} \tag{3}$$

Regardless of whether proportional damping exists or not, both σ_j and ω_j can be measured. However, rather large uncertainties are usually associated with measurements of σ_j which in turn reflect upon ζ_j. Corresponding pairs of ζ_j and ω_{0_j} can be plotted as in Fig. 1 with uncertainty bands indicated by the vertical line segments. If proportional damping exists, a curve corresponding to Eq. (3) could be passed through all of the uncertainty bands as indicated by the dotted line in Fig. 1. Otherwise nonproportional damping would be concluded. Although this is interpreted as a necessary condition for establishing proportional damping, it is not a sufficient condition. If it were, one would be led to conclude, for example, that all two degree-of-freedom systems have proportional damping since there are two arbitrary constants in Eq. (3) and such a curve can be fitted through any two points. Thus it should be possible to establish quite easily when nonproportional damping is present.

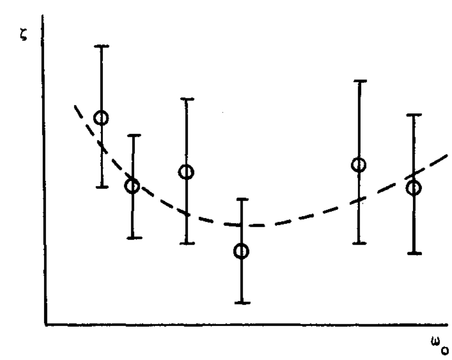

Fig. 1 A necessary condition for proportional damping.

A Perturbation Method for Calculating the Full Modal Damping Matrix

For linear systems with viscous damping, the n homogeneous equations of motion may be written

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{4}$$

where c denotes the viscous damping matrix. Transformation of these equations to the undamped modal coordinates q results in

$$\phi_R{}^T m \phi_R \ddot{q} + \phi_R{}^T c \phi_R \dot{q} + \phi_R{}^T k \phi_R q = 0$$

or

$$M\ddot{q} + C\dot{q} + Kq = 0 \tag{5}$$

The matrices M and K will be diagonal while in general C